Manifolds & Fibre Bundles

Manifolds

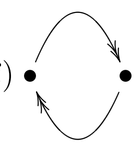
Recall that a manifold is a topological space that is locally Euclidean. We make this notion precise in the following definition.

(0.1) Definition: A *topological manifold of dimension* n consists of a pair $(X, \mathcal{A})$, where X is a Hausdorff, second-countable topological space, and $\mathcal{A}$—called an *atlas of charts* for X—is a collection of *coordinate charts*, which in turn are given by pairs (U, φ) with $U \subseteq X$ open and $\varphi : U \to V$ a homeomorphism for some open, contractible $V \subseteq \mathbb{R}^n$, with the property that $\mathcal{A}$ covers X—i.e., $\forall x \in X,\ \exists\,(U, \varphi) \in \mathcal{A} \text{ s.t. } x \in U$. By default, we assume $\mathcal{A}$ to be maximal—that is, it contains all possible coordinate charts for X—and suppress it from the notation.

Non-examples

(i) Γ

(ii) The Long Line

(iii) 

If (U, φ) is a coordinate chart, then φ gives a coordinate representation for points in U, and the pair (U, φ^{-1}) is called a *coordinate patch*. The upshot of working with manifolds is that we can use coordinates when working locally. Our first theorem shows that $\mathcal{C}(X)$ looks locally like $\mathcal{C}(\mathbb{R}^n)$.

(0.2) Theorem: (*Partition of Unity*) If X is a manifold and $\mathcal{U}$ is an open cover of X, then there exists a countable, locally-finite cover $\{U_i\}$ of X consisting of open subsets of elements of $\mathcal{U}$ and maps $\varphi_i : U_i \to [0,1]$, where φ_i is supported in a compact subset of U_i, and for each $x \in X$, we have $\sum_{x \in U_i} \varphi_i(x) = 1$.

(0.3) Corollary: (*Extending local maps*) Let X be a manifold and let $\varphi : \mathbb{R}^n \to U$ be a coordinate patch in X. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. Then $f \circ \varphi^{-1} : U \to \mathbb{R}$ is continuous, and the restriction $f \circ \varphi^{-1}\big|_C$ to any compact subset $C \subseteq U$ may be extended to a (continuous) map $\tilde{f} : X \to \mathbb{R}$, where $\tilde{f}\big|_C = f \circ \varphi^{-1}\big|_C$, and $\tilde{f}_{X \setminus U} \equiv 0$.

(0.4) Observation: The collection of topological manifolds forms a subcategory of Top called $\mathcal{C}^0$. In particular, we construct $\mathcal{C}^0$ by restricting our objects to manifolds and keeping the morphisms of the topological category—i.e., all continuous maps. It is easy to see that continuous maps preserve the structure of manifolds—in other words, that a map between manifolds is continuous if and only if it is continuous in local cooradinates around each point.

Short Exact Sequences

(0.5) Definition: Let A, B, C be groups, and let $\varphi : A \to B$ and $\psi : B \to C$ be homomorphisms. The sequence $\mathbb{1} \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to \mathbb{1}$ is called a *short exact sequence* of groups if it is *exact* at each of A, B, C, meaning that the image of the map from the left is equal to the kernel of the map to the right. In particular, φ must be injective, ψ must be surjective, and in B, $\ker\psi = \operatorname{Im}\varphi$. By the FFTA, $C \cong B/\ker\psi = B/\operatorname{Im}\varphi$ "$\cong B/A$." We will call $\mathbb{1} \to \ker\varphi \to G \to \operatorname{Im}\varphi \to \mathbb{1}$ the *canonical short exact sequence* associated to a group G and a homomorphism φ defined on G.

(0.6) Theorem: If there is a *splitting* of the short exact sequence in **(0.5)**—that is, if $\exists\, \zeta : C \to B$ a homomorphism with $\psi \circ \zeta = \mathbb{1}_C$—then we have $B = \operatorname{Im}\varphi \rtimes \operatorname{Im}\zeta \cong A \rtimes C$. In such a case, we may say that $\operatorname{Im}\zeta$ is a *subgroup complementary to* $\operatorname{Im}\varphi$ in B, and it is unique up to conjugacy by elements of $\operatorname{Im}\varphi$. In particular, if B is Abelian, then $B \cong A \times C$, and $\operatorname{Im}\varphi$ and $\operatorname{Im}\zeta$ may be said to be *orthogonal* in B where no ambiguity is entailed in doing so.

Fibre Bundles

(1.1) Definition: A *rough fibre bundle* is given by a quadruple (F, M, X, π), where F, M, X are topological spaces, $\pi : M \to X$ is a surjective map, and for each $x \in X$, we have a map $\iota_x : F \hookrightarrow M$ with $F \underset{\iota_x}{\cong} \pi^{-1}(\{x\}) = \operatorname{Im}\iota_x$. In this case, X is called the *base space*; F is called the *fibre*; π is called the *bundle projection*; and M, called the *total space*, is said to be a *(rough) F-bundle over X.*

(1.2) Definition: If we relax the condition from **(1.1)** that ι_x be a homeomorphism onto its image and instead insist only that it be a homotopy equivalence—i.e., that $\exists f : \pi^{-1}(\{x\}) \to F$ with $f \circ \iota_x \simeq \mathbb{1}_F$ and $\iota_x \circ f \simeq \mathbb{1}_{\pi^{-1}(\{x\})}$—then we say that *$M$ fibres over X with a fibre type of F* or simply M is an *F-fibration over X.*

(1.3) Proposition: If X and Y are manifolds and $f : X \to Y$ is an injective map, then f is called a *topological embedding*. A topological embedding is automatically a homeomorphism onto its image, and so the last condition of **(1.1)** is unnecessary if F and M are both manifolds.

Proof: By picture.

(1.4) Definition: A *(topological) fibre bundle* is a rough fibre bundle that is *locally trivial*. In particular, if (F, M, X, π) has the structure of a rough fibre bundle, we require an atlas of charts $\mathcal{A}$ consisting of pairs (U, φ), called *local trivialisations* satisfying:

(i) $U \subseteq X$ is open;
(ii) $\{U \mid (U, \varphi) \in \mathcal{A}\}$ is an open cover of X;
(iii) $\varphi : \pi^{-1}(U) \to U \times F$ is a homeomorphism; and
(iv) $\hat{\pi} \circ \varphi = \pi\big|_{\pi^{-1}(U)}$, where $\hat{\pi} : U \times F \to U$ is the usual projection.

$$\begin{array}{ccc} F & \hookrightarrow & M \\ & & \downarrow \pi \\ & & X \end{array}$$

$$F \hookrightarrow M \overset{\pi}{\twoheadrightarrow} X$$

$$1 \to F \to M \overset{\pi}{\to} X \to 1$$

The quintuple $(F, M, X, \pi, \mathcal{A})$ is now called a *fibre bundle*, and by abuse, M is called a fibre bundle (or an F-bundle) over X. We will again pass automatically to the maximal such $\mathcal{A}$ with respect to the containment lattice. Above and to the right are three distinct notational conventions for a fibre bundle; we will adopt the convention $F \hookrightarrow M \overset{\pi}{\twoheadrightarrow} X$.

Note that the maps ι_x from **(1.4)** are merely restrictions of our trivialisation maps to single fibres; thus, assuming the existence of the ι_x's is unnecessary in a direct definition of fibre bundles. We will be primarily concerned with the case where X is a manifold. By taking intersections, we can always arrange for our local trivialisations to occur within coordinate charts for X so that we can choose our trivialisations to have range $\mathbb{R}^n \times F$. In particular, if $F^m \hookrightarrow M \overset{\pi}{\twoheadrightarrow} X^n$, then for $x \in M$, choose a local trivialisation (U, φ) with $\pi(x) \in U$ so that $x \in \pi^{-1}(U)$, and $\varphi(x) \in \mathbb{R}^n \times F$. Passing to a coordinate chart around $\check{\pi} \circ \varphi(x) \in F$, we find that x has a neighbourhood homeomorphic to $\mathbb{R}^{n+m}$; therefore, M is a manifold of dimension $m + n$.

(1.5) Observation: Let $F \hookrightarrow M \overset{\pi}{\twoheadrightarrow} X$ be a fibre bundle as in **(1.4)**. If $(U, \varphi), (U', \varphi') \in \mathcal{A}$, the map $\varphi' \circ \varphi^{-1}\big|_{(U \cap U') \times F} : (U \cap U') \times F \to (U \cap U') \times F$ is called a *transition map* between the respective trivialisations, and $\hat{\pi} \circ \varphi' \circ \varphi^{-1}\big|_{(U \cap U') \times F} = \hat{\pi}\big|_{(U \cap U') \times F}$, $\forall a \in F$, and $\check{\pi}\varphi' \circ \varphi^{-1}\big|_{\{x\} \times F} \in \operatorname{Aut}(F)$, where $\check{\pi} : U \times F \to F$ is the projection map, and $\operatorname{Aut}(F)$ denotes the group of self-homeomorphisms of F. In particular, our transition maps have the form $\varphi' \circ \varphi^{-1}(x, a) = \big(x, [\xi(x)](a)\big)$, where $\xi : U \cap U' \to \operatorname{Aut}(F)$ is continuous. We will call ξ the *fibre transformation* associated to the transition map.

(1.6) Corollary: (*Deconstruction of fibre bundles*) In fact, **(1.5)** can be used to constructively generate fibre bundles in the following way. If X is any (locally compact) topological space and $\mathfrak{U}$ is a (locally finite) open cover of X, we construct F-bundles over X by setting $M = \left(\coprod_{U \in \mathfrak{U}} U \times F \right) \Big/ \sim$, with the relation $\sim$ being given by the transition maps. In particular, if $U, U' \in \mathfrak{U}$ and $x \in U \cap U'$, then $U \times F \ni (x, a) \sim \big(x, [\xi(x)](a)\big) \in U' \times F$, where ξ is again the fibre transformation map.

Examples

$$\begin{array}{ccc} [0,1] & \hookrightarrow & \mathbb{T}\times[0,1] \\ & & \downarrow\!\!\downarrow \\ & & \mathbb{T} \end{array} \qquad\qquad \begin{array}{ccc} [0,1] & \hookrightarrow & \ddot{M} \\ & & \downarrow\!\!\downarrow \\ & & \mathbb{T} \end{array}$$

$$\begin{array}{ccc} \mathbb{T} & \hookrightarrow & \mathbb{T}^2 \\ & & \downarrow\!\!\downarrow \\ & & \mathbb{T} \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{T} & \hookrightarrow & K \\ & & \downarrow\!\!\downarrow \\ & & \mathbb{T} \end{array}$$

(1.7) Remark: A fibre bundle $F \hookrightarrow M \twoheadrightarrow X$ is called *trivial* if $M \cong X \times F$. This serves as a post-hoc justification for the term "local trivialisation," but moreover stems from the deeper fact that in the construction **(1.6)**, the choice $\xi \equiv \mathbb{1}_{\mathrm{Aut}(F)}$ yields a trivial bundle. Given a cover of X by trivialisations, let G denote the subgroup of $\mathrm{Aut}(F)$ generated by the images of the maps ξ associated to the transition maps. G is called the *structure group* of $F \hookrightarrow M \twoheadrightarrow X$, and we can contruct a bundle $G \hookrightarrow E \twoheadrightarrow X$ whose transition maps are given by left translation by $\xi(x) \in G$ in the second component via a procedure parallel to the deconstruction/reconstruction of **(1.6)**. The bundle just constructed is known as the *principle G-bundle associated to the bundle $F \hookrightarrow M \twoheadrightarrow X$*. We'll revisit principal bundles more generally after a brief discussion of vector bundles.

(1.8) Definition: A fibre bundle $V \hookrightarrow E \twoheadrightarrow X$ is called a *vector bundle* if its fibre, V, has the structure of a vector space and its structure group is a subgroup of $\mathrm{GL}(V)$. If $V \hookrightarrow E \overset{\pi}{\twoheadrightarrow} X$ and $W \hookrightarrow F \overset{\tilde{\pi}}{\twoheadrightarrow} X$ are two vector bundles, and $\varphi : E \to F$, then φ is called a *map of vector bundles over X* if $\pi = \tilde{\pi} \circ \varphi$ and $\varphi\big|_{\pi^{-1}(\{x\})}$ is a linear map $\forall x \in X$. More generally, if $F \hookrightarrow M \overset{\pi}{\twoheadrightarrow} X$ and $E \hookrightarrow N \overset{\tilde{\pi}}{\twoheadrightarrow} Y$, a ***bundle map*** is a pair (φ, ψ) with $\varphi : M \to N$, and $\psi : X \to Y$ such that $\psi \circ \pi = \tilde{\pi} \circ \varphi$, which is to say that bundle maps take fibres to fibres and preserve the bundle projection maps. Now, a *map of vector bundles* is a bundle map, (φ, ψ), between vector bundles which is linear on fibres—i.e., $\varphi\big|_{\pi^{-1}(\{x\})} : \pi^{-1}(\{x\}) \to \tilde{\pi}^{-1}(\{\psi(x)\})$ is a linear transformation $\forall x \in X$.

Example

$$\begin{array}{ccc} \mathbb{R} & \hookrightarrow & \ddot{M} \\ & & \downarrow \\ & & \mathbb{T} \end{array}$$

(1.9) Observation: The collection of fibre bundles comprise the objects of a category $\mathcal{FB}$ whose morphisms are bundle maps. Moreover, vector bundles with vector budle maps form a subcategory $\mathcal{VB}$ of $\mathcal{FB}$. Furthermore, principal bundles also form a subcategory of $\mathcal{FB}$, where the morphisms are bundle maps which preserve the group action on the fibres. In order to further examine this, we formally introduce group actions.